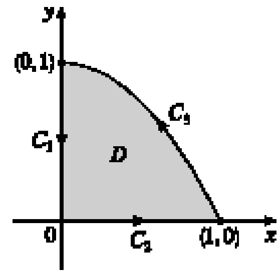
13.4: 4, 12, 14, 16, 20* (graph optional)

4. (a) $C_1: x = 0 \quad \Rightarrow \quad dx = 0\,dt,\ y = 1 - t \quad \Rightarrow$
$dy = -dt,\ 0 \le t \le 1$
$C_2: x = t \quad \Rightarrow \quad dx = dt,\ y = 0 \quad \Rightarrow \quad dy = 0\,dt,\ 0 \le t \le 1$
$C_3: x = 1 - t \quad \Rightarrow \quad dx = -dt,\ y = 1 - (1-t)^2 = 2t - t^2 \quad \Rightarrow$
$dy = (2 - 2t)\,dt,\ 0 \le t \le 1$



Thus

$$\oint_C x\,dx + y\,dy = \oint_{C_1+C_2+C_3} x\,dx + y\,dy$$
$$= \int_0^1 (0\,dt + (1-t)(-dt)) + \int_0^1 (t\,dt + 0\,dt) + \int_0^1 ((1-t)(-dt) + (2t - t^2)(2-2t)\,dt)$$
$$= \left[\tfrac{1}{2}t^2 - t\right]_0^1 + \left[\tfrac{1}{2}t^2\right]_0^1 + \left[\tfrac{1}{2}t^4 - 2t^3 + \tfrac{5}{2}t^2 - t\right]_0^1$$
$$= -\tfrac{1}{2} + \tfrac{1}{2} + \left(\tfrac{1}{2} - 2 + \tfrac{5}{2} - 1\right) = 0$$

(b) $\oint_C x\,dx + y\,dy = \iint_D \left[\dfrac{\partial}{\partial x}(y) - \dfrac{\partial}{\partial y}(x)\right] dA = \iint_D 0\,dA = 0$

12. $\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y)\right] dA = \iint_D (\cos y - \cos y)\,dA = \iint_D 0\,dA = 0$

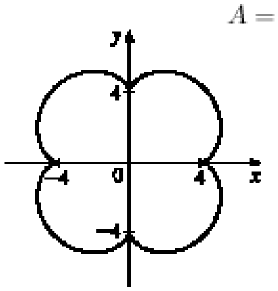
14. $\mathbf{F}(x,y) = \langle y^2\cos x, x^2 + 2y\sin x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C}(y^2\cos x)\,dx + (x^2 + 2y\sin x)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}(x^2 + 2y\sin x) - \tfrac{\partial}{\partial y}(y^2\cos x)\right] dA$$
$$= -\iint_D (2x + 2y\cos x - 2y\cos x)\,dA = -\int_0^2 \int_0^{3x} 2x\,dy\,dx$$
$$= -\int_0^2 2x\,[y]_{y=0}^{y=3x}\,dx = -\int_0^2 6x^2\,dx = -2x^3\big]_0^2 = -16$$

16. $\mathbf{F}(x,y) = \left\langle y - \ln(x^2+y^2), 2\tan^{-1}\left(\dfrac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at $(2,3)$. C is oriented positively, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y - \ln(x^2+y^2))\,dx + \left(2\tan^{-1}\left(\frac{y}{x}\right)\right)dy = \iint_D \left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}(y - \ln(x^2+y^2))\right] dA$$
$$= \iint_D \left[2\left(\frac{-yx^{-2}}{1 + (y/x)^2}\right) - \left(1 - \frac{2y}{x^2+y^2}\right)\right] dA = \iint_D \left[-\frac{2y}{x^2+y^2} - 1 + \frac{2y}{x^2+y^2}\right] dA$$
$$= -\iint_D dA = -(\text{area of } D) = -\pi$$

20.



$$A = \oint_C x\,dy = \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$
$$= \int_0^{2\pi} (25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$
$$= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi}$$

[Use Formula 80 in the Table of Integrals]

$$= 30\pi$$